# Funky FAMAT Manual

October 8, 2020

# Algebra 2

## (1) Inverse of Rational Expressions

**Q:** The function $y = g(x)$ has the property that $f(g(x)) = g(f(x)) = x$. If $f(x) = \frac{2x-1}{x+1}$ and $g(x) = \frac{x+a}{b+cx}$ then give the value of $(a + b + c)$.

**A:** $f(g(x)) = g(f(x)) = x$ denotes that $f(x)$ and $g(x)$ are inverse function. As always with inverse functions, the first step is to switch x and y, so the equation is $x = \frac{2y-1}{y+1}$. In order to solve for $y$, the denominator is moved to the left side through multiplication, making the equation $x*(y+1) = 2y-1$. The x is distributed, making the expression $xy+x = 2y-1$. Moving all the terms with y to one side and terms without y to the other, $2y$ is subtracted and moved to the left side and $x$ to the right side, making the equation $xy - 2y = -1 - x$. Now, a $y$ can be factored out, so $y(x - 2) = -1 - x$. Finally, to isolate y, $(x - 2)$ is divided on both sides, making the equation $y = \frac{-1-x}{x-2}$, which is also $y = \frac{1+x}{2-x}$. Thus, $a = 1$, $b = 2$, and $c = -1$. So, $a + b + c = 1 + 2 + -1 = 2$.

*[March Regional 2020]*

## (2) U-Substitution

**Q:** The equation $(3x - 5)^{\frac{4}{3}} + 5(3x - 5)^{\frac{2}{3}} = 36$ has real solutions $r$ and $s$ such that $r > s$. Give the value of $3(r - s)$.

**A:** At first look, this equation looks difficult to solve with normal methods of re-arranging and factoring. There is a common term of $(3x - 5)^{\frac{2}{3}}$, so an arbitrary variable, u, can be set equal to $(3x - 5)^{\frac{2}{3}}$, so $u = (3x - 5)^{\frac{2}{3}}$. Plugging $u$ in, the equation is $u^2 + 5u = 36$ (note that $x^{\frac{4}{3}}$ is $x^{\frac{2}{3}^2}$. Now, this can be solved like a normal quadratic. Moving 36 to the left side and factoring, the equation is $(u + 9)(u - 4)$. Setting each factor equal to 0, the solutions are $u = -9$ and $u = 4$. Re-substituting $(3x - 5)^{\frac{2}{3}}$ for $u$, the solutions are $(3x - 5)^{\frac{2}{3}} = -9$ and $(3x - 5)^{\frac{2}{3}} = 4$. Now, x must be solved for. Raising both sides to $\frac{3}{2}$, the equations are $3x - 5 = \pm 8$ and $3x - 5 = \pm 27i$. The second equation contains imaginary solutions, and the question asks for real solutions, so it is thrown out. Isolating x in the first one by adding 5 and dividing by 3, the solutions are $x = -1$ and $x = \frac{13}{3}$. Thus, $3(r - s) = 3(\frac{13}{3} - (-1)) = 16$.

*[#17 March Regional 2019]*

## (3) U-Substitution with Logarithms

**Q:** Let the sum of the solutions of $(ln(x))^2 - 4ln(x) + 3 = 0$ be $E$. Find the greatest integer less than or equal to $E$ (Note that $e = 2.7, e^2 = 7.4, e^3 = 20.1$).

**A:** Again, noting that this equation resembles a quadratic but with $ln(x)$, we can state that $u = ln(x)$, making the equation $u^2 - 4u + 3 = 0$. This is easily factorable into $(u - 3)(u - 1) = 0$. Setting each factor equal to 0, $u = 3$ and $u = 1$. Re-substituting $ln(x)$ for $u$, we have $ln(x) = 3$ and $ln(x) = 1$. Solving for $x$ by raising both sides using $e$, we have that $e^{ln(x)} = e^3$ and $e^{ln(x)} = e^1$. Thus, $x = e^3$ and $x = e$, which according to the given approximations in the question, $e = 2.7$ and $e^3 = 20.1$. The sum of these solutions is $20.1 + 2.7 = 22.8$ and so, the greatest integer less than or equal to $E$ is 22.

*[#12 January Regional 2015]*

## (4) Systems of Equations with Non-Linear Variables

**Q:** Find the sum of all y-coordinates of the solutions of the following system: (1) $4x + y - 3 = x^2$ and (2) $3x - y = 7$.

**A:** Although there's an $x^2$ present, equations like these can be solved like a normal (linear) systems of equations by using substitution. Solving for $y$ in equation (2), we have $3x - 7 = y$. Plugging this equation into (1), we have $4x + (3x - 7) - 3 = x^2$. Combining like terms and moving all the terms to the right side, we have $x^2 - 7x + 10$. This is easily factorable into $(x - 5)(x - 2)$ and so $x = 2$ and $x = 5$. Plugging these back into equation (2) to get the y-coordinates, we have $3(2) - y = 7$, or $y = -1$ and $3(5) - y = 7$, or $y = 8$. Thus, the sum of the y-coordinates is $8 + (-1) = 7$.

*[#1 March Invitational 2011]*

## (5) Manipulating Quadratics

**Q:** If $2x^2 + 6x - 2 = 0$, find the value of $x - \frac{1}{x}$.

**A:** Algebra 2 classes teach that the only ways to solve quadratics are to factor or use the quadratic formula, but quadratics are actually quite versatile and can be manipulated in many ways. Looking at this equation, we see immediately that we can factor out a 2, simplifying the equation to $x^2 + 3x - 1 = 0$. Now, we could choose to use the quadratic formula, but finding $x - \frac{1}{x}$ would be a pain. Instead, if we divide by x, note that we get $x + 3 - \frac{1}{x} = 0$, which more closely resembles what the question is asking for. Isolating the variables, we get $x - \frac{1}{x} = -3$, which happens to be our answer. Thus, without finding the actual x-intercepts of this equation, we have found that $x = \frac{1}{x} = -3$.

*[#29 February Invitational 2012]*

## (6) Manipulating Rational Expressions

**Q:** If $x^2 + \frac{1}{x^2} = 3$, then find $x^3 + \frac{1}{x^3}$.

**A:** This takes a few steps. First, let's find what $x + \frac{1}{x}$ equals because this is easier to manipulate and find an $x^3$ term. Thus, starting with the equation $x + \frac{1}{x} = C$, we square this to get $x^2 + 2 + \frac{1}{x^2} = C$. Therefore, $x^2 + \frac{1}{x^2} = C^2 - 2$. We know that $x^2 + \frac{1}{x^2} = 3$ as given by the problem, so $C^2 - 2 = 3$ and $C = \sqrt{5}$.
Now, we can multiply $x + \frac{1}{x}$ by $x^2 + \frac{1}{x^2}$, which yields $x^3 + \frac{1}{x^3} + x + \frac{1}{x} = 3 * \sqrt{5}$. Plugging in $\sqrt{5}$ for $x + \frac{1}{x}$, we have that $x^3 + \frac{1}{x^3} = 3\sqrt{5} - \sqrt{5} = 2\sqrt{5}$.

*[#9 January Invitational 2019]*

## (7) Vieta's Formula

**Q:** Find the sum of the square roots of the polynomial $f(x) = 2x^2 + 6x + 6$

**A:** While we could find the roots and square them using the quadratic formula, an easier way would be utilizing Vieta's Formulas. Specifically, the sum of roots being $\frac{-b}{a}$ and the product of the roots $\frac{c}{a}$. Designating the roots as $r_1$ and $r_2$, we see that $(r_1 + r_2)^2 = r_1^2 + 2r_1r_2 + r_2^2$. Because we want to find what $r_1^2 + r_2^2$ is, we will isolate it by moving the $2r_1r_2$ term to the left side, making the expression $(r_1 + r_2)^2 - 2r_1r_2 = r_1^2 + r_2^2$. Knowing that $r_1 + r_2 = \frac{c}{a} = \frac{6}{2} = 3$ and $r_1 + r_2 = \frac{-b}{a} = \frac{-6}{2} = -3$, we can plug in these values into our expression. Thus, $(-3)^2 - 2(3) = 3 = r_1^2 + r_2^2$. The sum of the square of the roots is 3.

*[Original]*

## (8) Trailing Zeroes

**Q:** Find the number of trailing zeroes in the expansion of 9743!

**A:** Obviously, 9743! is way too large of a number to expand by hand. However, knowing that a trailing zero is created whenever a number is multiplied by 10 (note that a trailing zero is a zero at the end of a number, so $10 * 93$ has 1 trailing zero), if we find the number of 10's that are multiplied within 9743!, we can find the number of trailing zeros. 10 is composed of 5*2, so if we find the number of factors of 5's and 2's in 9743!, we'll find the number of 10's that are formed and thus, trailing zeros. There are fewer factors of 5 than 2 in 9743!, so we will only count factors of 5, since that will be the limiting factor.
$\lfloor \frac{9743}{5} \rfloor = 1948$. We must also test for $5^2$, $5^3$, $5^4$, and $5^5$ because we only count 1 factor of 5 initially when dividing by $5^1$ (25 contains 2 factors of 5), so $\lfloor \frac{9743}{5^2} \rfloor = 389$, $\lfloor \frac{9743}{5^3} \rfloor = 77$, $\lfloor \frac{9743}{5^4} \rfloor = 15$, and $\lfloor \frac{9743}{5^5} \rfloor = 3$. Adding these up, we have $1948 + 389 + 77 + 15 + 3 = 2432$ factors of 5 in 9743. Thus, there are 2432 trailing zeros in the expansion of 9743!.

*[#21 March Invitational 2019]*

## (9) Direct Variation

**Q:** The number of dates Ali eats in the evening varies directly with the number of hours he works during the day. When Ali works for 8 hours, he eats 5 dates. How many dates will Ali eat if he works for 7 hours? (Ali can eat fractions of a date.)

**A:** Direction variation is where the y-variable varies by rate k with the x-variable, otherwise written as $y = kx$. Plugging in the given values for y and x where hours worked are x and # of dates eaten are y, $5 = k(8)$, $\frac{5}{8} = k$. Thus, if Ali works for 7 hours, $y = \frac{5}{8}(7)$, so $y = \frac{35}{8}$.

*[#12 January Regional 2017]*

## (10) Infinite Radicals

**Q:** Evaluate: $2 * \sqrt[3]{24 + \sqrt[3]{24 + \sqrt[3]{24 + ...}}}$

**A:** Nearly every single infinite radical problem given by FAMAT can be solved by substituting $x$. Because the radical goes on for infinity, we can state that $x = 2 * \sqrt[3]{24 + \sqrt[3]{24 + \sqrt[3]{24 + ...}}}$, and so we can both substitute x within the equation as well as set the equation equal to x. Thus, $x = \sqrt[3]{24 + x}$. We can remove the factor of 2 in front entirely because we are working with infinities and 2 is very small comparable. Now, this can be solved like a normal one-variable equation. Dividing by 2 and cubing both sides, we have $x^3 = 24 + x$. Moving all the x-terms to the left side, we have $x^3 - x = 24$. While solving cubics can be difficult, by noting that $3^3$ is the smallest cube greater than 24, we can substitute it, finding that it does work and thus, $x = 3$.

*[#20 January Regional 2019]*

## (11) Sum of Coefficients

**Q:** What is the sum of the coefficients of $(x + 4)^5$ when it is expanded?

**A:** Although one may be tempted to expand the polynomial in order to add the coefficients, a much easier method is to simply substitute 1. Noting that plugging in 1 simply eradicates any effect that the variable has, leaving the coefficients behind in an expression to be added, we can plug-in 1 into the factored form to find the sum of the coefficients.

Thus, $(1+4)^5 = 5^5 = 3125$, the sum of the coefficients is 3125.

*[#29 January Regional 2019]*

## (12) Distinguishable Permutations of a Word

**Q:** Which of the following expressions in equivalent to the number of distinguishable permutations of "LAWTONCHILES"?

**A:** There are 12 letters in LAWTONCHILES and thus, 12! ways to rearrange the word. However, there are 2 L's and thus, 2! versions of the permutations will be repeated. Therefore, the number of *distinguishable* permutations of LAWTONCHILES is $\frac{12!}{2!}$.

*[#28 January Invitational 2020]*

## (13) Binomial Theorem

**Q:** What is the coefficient of the $x^{15}$ term in the expansion of $(\pi x - 7)^{83}$?

**A:** Attempting to expand this binomial would be a challenge, so instead, we'll make use of binomial theorem. Essentially, we know that when a binomial is expanded, it is in the form of $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n}x^0 y^n$. There's clearly a pattern where the coefficient is comprised of $\binom{n}{m}$ where m is the exponent on the 2nd term, and the exponents on the two terms add to $n$.
Thus, we know that when $x^{15}$ occurs, the $\pi x$ term is being raised to the 15th power. Because the exponents must sum to 73, the exponent on the $-7$ term is $83 - 15 = 68$, so $-7^{68}$ or simply $7^{68}$. We know the two variable terms are in the form of $(\pi x)^{15}7^{68}$, so now we just need to solve the coefficient. Because the exponent on the 2nd term is 68, the coefficient is $\binom{83}{68}$ (note that $n = 83$ and $m = 68$) and so, our final answer is that the coefficient of the $x^{15}$ term is $\binom{83}{68}(\pi)^{15}7^{68}x^{15}$.

*[#13 March Regional 2018]*

## (14) Converting Repeating Decimals

**Q:** What is the fractional representation of the repeating decimal $2.\overline{365}$?

**A:** Let's simply focus on the decimal part and assign the variable $x = 0.\overline{365}$. The decimal repeats every 3 places, so we'll multiply $x$ by 1000 to get $1000x = 365.\overline{365}$. Subtracting $x$ from $1000x$, $1000x - x = 365.\overline{365} - 0.\overline{365}$, which evaluates to $999x = 365$. Solving for $x$, we have $x = \frac{365}{999}$. Thus, the fractional representation of $0.\overline{365}$ is $\frac{365}{999}$ meaning that $2.\overline{365}$ can be represented by $365 + 2 * 999 = \frac{2363}{999}$.

*[Original]*

## (15)

**Q:** Which of the following is equivalent to $|4 + 8i|$?

**A:** In order to find the absolute value of an imaginary expression, we find the sum of the square of the real and imaginary parts and take the square root of that. Thus, the absolute value of $4 + 8i$ is $4^2 + 8^2 = 80$, and the square root of 80 is $4\sqrt{5}$.

*[#7 February Regional 2021]*

## (16) Polynomial Remainders

**Q:** What is the remainder when $2x^3 + 7x^2 - 3x + 6$ is divided by $(x + 3)$?

**A:** Now, we could go through the trouble of manually dividing the two expressions, but a much easier method is to use synthetic division. Polynomial Remainder Theorem states that if a polynomial $f(x)$ is divided by a factor $(x - a)$, the remainder is equal to a constant $f(a)$. The $a$ in this case is $-3$, so plugging in $f(-3)$, we get $f(-3) = 24$. Thus, the remainder is simply 24.

*[#11 February Regional 2021]*

## (17) Manipulating Logarithms & Exponents

**Q:** Solve for $x : 8^{\log_8 (x\sqrt{32})} = 8\sqrt{2}$

**A:** At first glance, this looks overly complicated. However, referring to properties of logarithms, it is known that $a^{\log_a b} = b$, as the exponent and logarithm cancel each other out. Therefore, the problem simplifies to just $x\sqrt{32} = 8\sqrt{2}$. Simplifying $\sqrt{32}$ by pulling out a 4, the problem becomes $4x\sqrt{2} = 8\sqrt{2}$. Dividing by $sqrt2$, the problem is left as $4x = 8$. Thus, $x = 2$.

*[#14 February Regional 2021]*